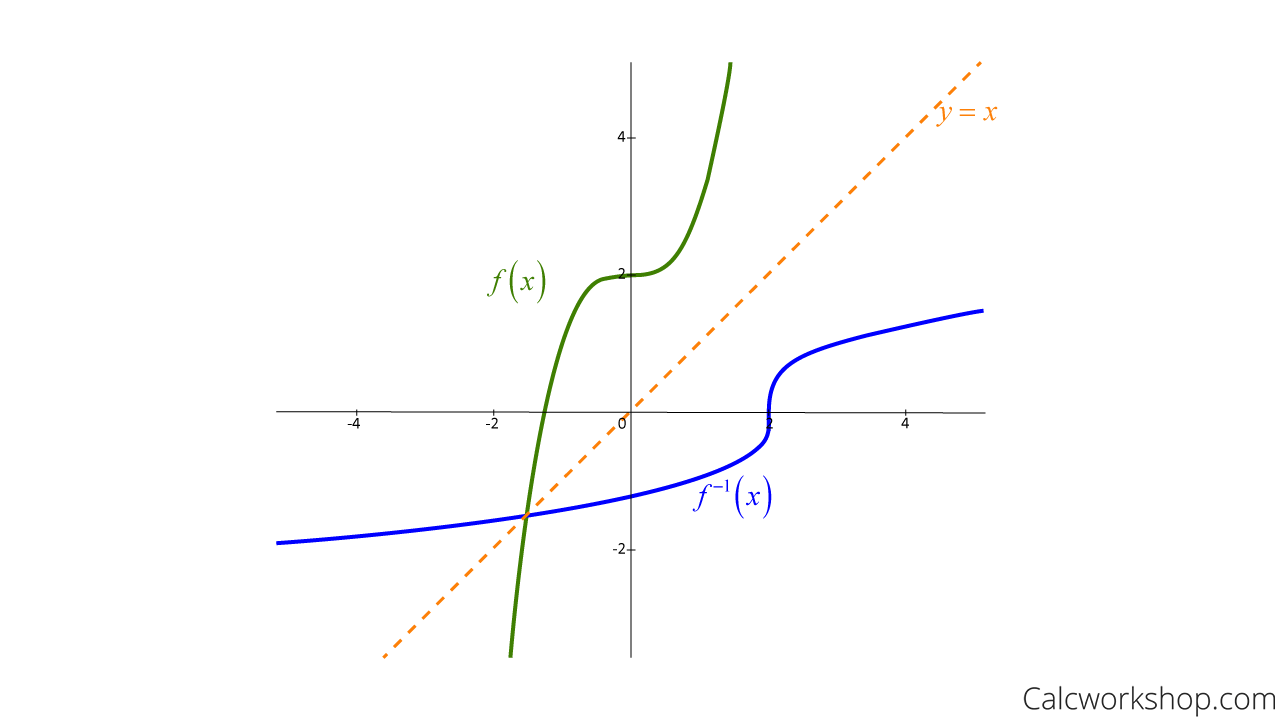
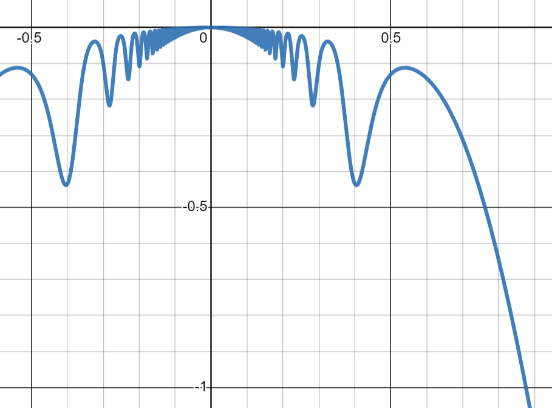
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**MAT137: Calculus with Proofs**

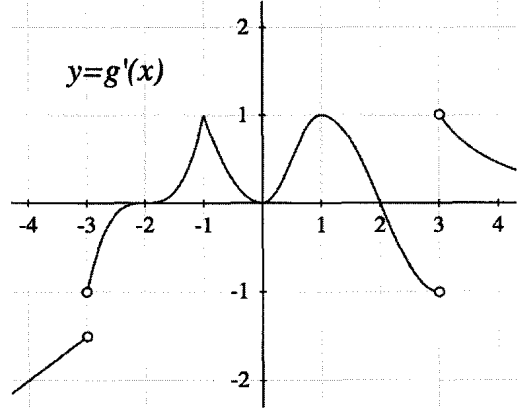
A (Hopefully) Concise Guide

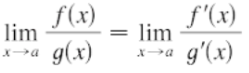


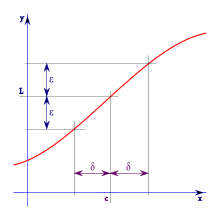
By Richard Yin

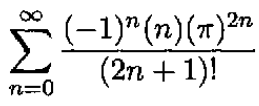
University of Toronto

Based on content from Fall-Winter 2021-2022









# Preface

These notes are UTSG-centered and based off notes and remarks (which I may not remember correctly) by my MAT137 professor, Xiao Jie, who made a course of suffering more bearable. The purpose of this textbook is to **be as dry and succinct as possible** as to not strain your eyes with paragraphs like most textbooks.

The accompanying document, *Questions in MAT137*, has practice questions (some from past exams) with partial solutions. It is recommended you check them out.

**Choosing a Calculus Course**

All first-year calculus courses are full-credit (Y) or two half-credit (F/S) courses. Consider [this](https://www.math.toronto.edu/cms/undergraduate-program/current-students-ug/guide-1st-year-calculus/) guide.

|  |  |  |
| --- | --- | --- |
| **Course** | **Recommended For** | **Description** |
| [MAT133](https://artsci.calendar.utoronto.ca/course/mat133y1) | Commerce | Commerce-focused, easy and shallow. Crams calculus AND linear algebra (1.0 + 0.5 credits) into a 1.0 credit course. Not a valid prerequisite for all second-year math courses. |
| [MAT135](https://artsci.calendar.utoronto.ca/course/mat135h1)/  [MAT136](https://artsci.calendar.utoronto.ca/course/mat136h1) | Science  Social Sciences | Split into derivatives (MAT135) and integrals (MAT136). The standard calculus courses. Mostly applied math, no theory. Computation, modelling, and application-focused. |
| [MAT137](https://artsci.calendar.utoronto.ca/course/mat137y1) | Computer Science  Physics  Statistics  Actuarial Science  Economics | A combination of theory and computation, balancing the extremes of MAT135/136 and MAT157. Introduces mathematical rigour, definitions, and proofs, whose initial learning curve is very steep. |
| [MAT157](https://artsci.calendar.utoronto.ca/course/mat157y1) | Math Specialists | Mostly theory, the most difficult one. Delves into complex proofs & abstract mathematics. “[Analysis](https://en.wikipedia.org/wiki/Mathematical_analysis)” is essentially advanced calculus. |

There’s also MAT134 (a UTM thing, similar to MAT135/136). Here’re some course-specific content:

|  |  |
| --- | --- |
|  | MAT135/136 computations are hard. Unique concepts include, probability models, and differential equations.  But…you can just formally study differential equations in [MAT244](https://artsci.calendar.utoronto.ca/course/mat244h1) and probability in STA[237](https://artsci.calendar.utoronto.ca/course/sta237h1)/[257](https://artsci.calendar.utoronto.ca/course/sta257h1). |
|  | MAT137’s computations generally *aren’t as hard* as 135/136. Exams have a **proofs section** worth roughly 25%.  There’re less word questions and application-based questions. |
|  | MAT157 is frightening. Proofs make up ~33% of the exam, and are **very abstract** – even the computations.  Computations often combine many concepts (limit, integral, derivative).  MAT157 unlocks the same courses as 137, plus a few unique extras, like:   * [MAT257 (Analysis II)](https://artsci.calendar.utoronto.ca/course/mat257y1), * [MAT267 (Advanced Ordinary Differential Equations)](https://artsci.calendar.utoronto.ca/course/mat267h1) * [MAT327 (Topology)](https://artsci.calendar.utoronto.ca/course/mat327h1)   Check out these 157 notes: [#1](https://www.math.toronto.edu/~drorbn/classes/0203/157AnalysisI/index.html), [#2](http://www.math.toronto.edu/karshon/courses/mat157/) |

Anything with MAT135/136 as a prerequisite accepts 137 too. Anything that accepts 137 also accepts 157.

You can take [MAT138 (Introduction to Proofs)](https://artsci.calendar.utoronto.ca/course/mat138h1) alongside these courses to get familiar with proofs, but I don’t recommend it – they’re so many other cool things you can learn in university!

If you didn’t take MAT157 but want to take courses only accepting MAT157 as prerequisite, usually [MAT246 (Concepts in Abstract Mathematics)](https://artsci.calendar.utoronto.ca/course/mat246h1) or [PHL245 (Modern Symbolic Logic)](https://artsci.calendar.utoronto.ca/course/phl245h1) can unlock some.

**Linear Algebra**

[MAT223 (Linear Algebra I)](https://artsci.calendar.utoronto.ca/course/mat223h1) is easier than MAT137 in my opinion. Its computations are dry and mechanical; the hard part is terminology and visualizing the problems.

[MAT224 (Linear Algebra II)](https://artsci.calendar.utoronto.ca/course/mat224h1) gets theoretical and is more comparable to MAT137.

[MAT240 (Algebra I)](https://artsci.calendar.utoronto.ca/course/mat240h1) and [MAT247 (Algebra II)](https://artsci.calendar.utoronto.ca/course/mat247h1) are the MAT157/257 equivalents of linear algebra, and even require taking 157/257 as corequisites.

Linear algebra is very important in statistics, modelling, AI and machine learning, economics, quantum mechanics, theoretical chemistry, etc. So if you’re going into those fields, at least consider MAT223!

**Advice**

1. This textbook probably teaches in a different order than your professor.
2. MAT137 is hardest in the first 2-3 months: the learning curve of grasping how limit proofs work.
3. For test/exam review, don’t review concepts; do practice questions. You learn math via example.
   * For proofs, it builds up an intuition of how to approach problems.
   * For computations, it helps you easily remember formulas.
4. If there’re questions that stump you, use [Desmos](https://www.desmos.com/calculator) and [Symbolab](https://www.symbolab.com/) if it’s computation. Otherwise, use office hours and tutorials. Or, if you’re in UTSG, use the [Math Learning Centers (MLC)](http://www.math.toronto.edu/cms/undergraduate-program/current-students-ug/math-learning-centres/).
5. In my opinion, the best predictor of university success is work ethic – studying the way you find most effective, and not finishing work the day before it’s due.

**About this Guide**

My credentials are: I’m a UTSG comp-sci student who got 95% on MAT137. In highschool, I took IB SL math.

Here’s a refresher on large operators. I will only use summation, but higher-level courses might use more.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

All example questions are valid exam-style questions unless otherwise stated. All content in this guide, except for some example questions, was written by me (though the solutions are original).

When I use “”, I’m probably paraphrasing my math prof. Credit goes where it’s due. I’m not monetizing this or anything so don’t sue me aaahh

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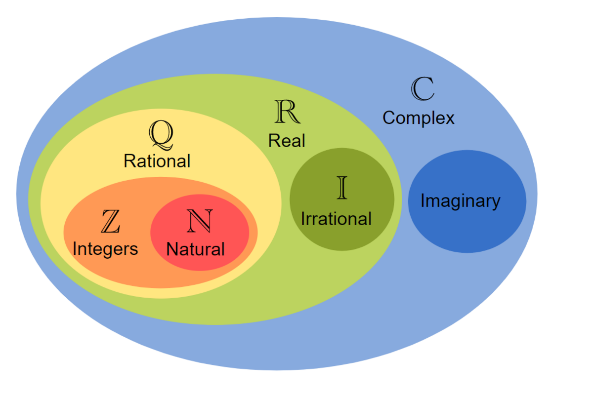
|  |  |
| --- | --- |
| 1. **Sets** 2. **Proofs**    1. Logical Symbols    2. Proof Structure    3. Negations and Proof by Contradiction    4. Proof by Induction 3. **Limits**    1. Theory       1. Definition ()       2. Notation       3. Absolute Value       4. Types of Limits       5. Proving a Limit       6. Proving a Limit ()       7. Disproving a Limit    2. Computation       1. General Methods       2. Limit Techniques 4. **Derivatives**    1. Theory       1. Definition as Limit       2. Definition as Slope       3. Definition in Kinematics       4. Derivative Properties    2. Computation       1. General Methods       2. L’Hôpital’s Rule       3. Logarithmic Differentiation       4. Implicit Differentiation    3. Application       1. Optimization       2. Related Rates 5. **Continuity**    1. Theory       1. Definition       2. Types of Discontinuities    2. Limit Laws    3. Differentiability and Continuity    4. Important Theorems       1. Intermediate Value Theorem (IVT)       2. Extreme Value Theorem (EVT)       3. Mean Value Theorem (MVT)       4. Proofs with Important Theorems | 1. **Inverse Functions**    1. Theory       1. Definition       2. Inverse Trigonometric Functions    2. Computation 2. **Graphing**    1. Simplifying Rational Fractions       1. Long Division       2. Partial Fractions    2. Basic Properties    3. Derivatives and Graphs    4. Asymptotes 3. **Integrals**    1. Theory       1. Definition as Riemann Sum       2. Definition of Integrability       3. Proving Integrability       4. Disproving Integrability       5. Definition as Supremum/Infimum       6. Fundamental Theorem of Calculus (FTC)       7. Integral Properties    2. Computation       1. General Methods       2. Trigonometric Integrals       3. Trigonometric Substitution       4. Improper Integrals    3. Application       1. Area       2. Volume 4. **Sequences and Series**    1. Sequences       1. Definition       2. Proving Convergence    2. Series       1. Definition       2. Convergence Tests       3. Power Series       4. Taylor Series       5. Telescoping Series 5. **Resources** |

# 

# 1. Sets

Sets are unordered collections of elements; in math, these elements are numbers. Sets are notated .

|  |  |
| --- | --- |
| **Notation** | **Meaning** |
|  | Empty set  *x* is an element of/in set *A*  **Complement** – The set of everything not in set *A*. If , then and vice versa  **Union** – The set containing everything in *A* or *B (inclusive “or”)*  **Intersection** – The set containing everything in *A* and *B*  **Cardinality** – The size/length/number of elements in set *A* |
|  | **Containment** – Set *A* is a **subset** of set *B* (everything in *A* is in *B*). and mean the same.  **Equality** – and (ie. A and B are identical) |

The above is essentially unused in MAT137; it’s more important for Linear Algebra (MAT223). However, the number sets below will be useful references:

|  |  |  |
| --- | --- | --- |
| **Set** | **Meaning** | **Example Subsets** |
|  | Complex number  Real number\*  Irrational number  Rational number *(writable as fraction)*  Integer  Natural number\*\* |  |

*\* (the default) refers to all numbers in 1D space. means all numbers in n-D space. MAT223 uses this concept*

*\*\*May/may not include 0 in math. Ask your math prof; I’ll say it does. Includes 0 in computer science*

Certain notations can describe more specific sets of numbers:

|  |  |  |
| --- | --- | --- |
| **Interval Notation** | **Set-Builder Notation** | **English Translation** |
|  |  | Set of real numbers between *a* (exclusive) and *b* (exclusive)  Set of real numbers between *a* (exclusive) and *b* (inclusive)  Set of real numbers greater or equal to *a* |

Set-builder notation, or , reads as “The set of all where .” can be wordy and include conjunctions (ie. *and*, *or*). This notation is flexible and multipurpose, but more common in MAT223.

Interval notation is simpler and used in MAT137. Intervals like (a, b) are **open**, [a, b] are **closed**, and [a, b) are **half-open.** Intervals like (a, ) are **unbounded** and aren’t very “proper”, mathematically speaking, due to the .Note that there must be “(“ or “)” next to .

The **maximum/minimum** is the largest/smallest item in a set.

* has a **maximum** (*b*) and **minimum** (*a*).
* has a **maximum** (*b*), but **no minimum**.

An **upper/lower bound** of a set is greater than or equal to/less than or equal to every set item.

* The **supremum** (sup) is the smallest possible upper bound.
  + has a **supremum** (*b*) and **infimum** (*a*).
* The **infimum** (inf) is the largest possible lower bound.
  + has a **supremum** (*b*) and **infimum** (*a*).

The *max/min* of a set is in the set. This isn’t necessarily true for *sup/inf*. You’ll find formal definitions soon.

# 2. Proofs

## 2.1. Logical Symbols

|  |  |  |
| --- | --- | --- |
| **Connective** | **Name** | **Meaning** |
| , , not \*  *,* | Negation  Conjunction  Disjunction  Implication/Conditional  Biconditional | *P* is not true  *P* is true and *Q* is true  *P* is true or (inclusive) *Q* is true  If *P* is true, *Q* istrue.  If and only if\*\* P is true, Q is true. (ie. and ) |

\**You see in CSC110/CSC165 and ~P in PHL245. My MAT137 prof didn’t use any symbols for not/and/or.*

\*\*”*If and only if” is often abbreviated to “iff”*.

|  |  |  |
| --- | --- | --- |
| **Quantifier** | **Name** | **Meaning** |
|  | Universal Quantifier  Existential Quantifier | For every/all/any *x* in set *S*, *P* is true  For some/there exists *x* in set *S,* (such that) *P* is true |

Below is a **truth table** containing all truth/false combinations and all connectives’ truth values.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  | \*This is called **vacuous truth**, where conditional is true since premise is false. You can’t prove with an example where is false. |
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | F | T | T\* | F |
| F | F | F | F | T\* | T |

## 2.2. Proof Structure

Think of proofs like using a **toolbox** ofknown things (in math, it’s stuff like or ), to show something new. The following tables show how to start proofs in the face of different statement types.

|  |  |  |  |
| --- | --- | --- | --- |
| **Implication** | **Conjunction** | **Disjunction** | **Biconditional** |
| eg. Prove | eg. Prove | eg. Prove | eg. Prove |
| Assume P  *Find a way to show Q* | *Separately show P*  *Separately show Q* | *Split into two cases:*  (1) Assume *P* is true  Thus is true  (2) Assume *P* is false  *Find a way to show Q* | Show  Assume *P*  *Find a way to show Q*  Show  Assume *Q*  *Find a way to show P* |
| Or, prove an equivalent form –  the **contrapositive:** |  | You can also use cases when *Q* is true/false. | Often seen in theorems and definitions. |

|  |  |
| --- | --- |
| **Universal Instantiation** | **Existential Instantiation** |
| eg. Prove | eg. Prove |
| Let  *Find a way to show* | Pick *something in S*  *Find a way to show* |
| We’re picking a random/arbitrary/any in , making no assumptions about what is, only assumptions about . If the proof still works despite the lack of specifics, we know the proof will work for all . | Since we’ve claimed is true for some , we have to back it up with a specific . Since we’ve chosen something specific, we can make assumptions based on what is. |

d

|  |  |  |
| --- | --- | --- |
| Note that |  | , but (. How come? Compare “For all *x*, there’s |

a negative *y = -x*” to the impossible “There’s a specific number *y* negative to all *x*. For all *x*, *y = -x*”.

|  |  |
| --- | --- |
| **Set Containment** | **Set Equality** |
| eg. Prove | eg. Prove |
| Let  *Find a way to show* | *Find a way to show*  *Find a way to show* |

Here are some rules of inference (from PHL245. Learning [derivations](https://youtu.be/6OoV1YyMP-Y?t=786) gives strong proofs foundations). These are examples of what you can conclude without proof based on given assumptions.

|  |  |
| --- | --- |
| **Logical Techniques** | **Name** |
|  | Modus ponens (mp)  Modus tollens (mt) – *based on proving the contrapositive*  Modus tollendo ponens (mtp)  Simplification (s)  Addition (add) |

Some examples of “theorems” you can use without proof are:

* – *similar logic for minimums*
* – *similar logic for floor*
* and – *similar logic for <*
* – *the Triangle Inequality*
* – *the Reverse Triangle Inequality*

## 2.3. Negations and Proof by Contradiction

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| If a statement is wrong, you show it by proving the statement’s negation true. When negating a statement, replace the symbols as shown to the right. Note swapping pairs and .  You must also invert equalities and inequalities | |  |  |  | | --- | --- | --- | |  |  |  | |  |  | |  |  | |  |  | |  |  | |  |  | |

|  |  |
| --- | --- |
| **Contradiction** | Proofs by contradiction require you assume a statement’s negation. They’re generally hard, as often, you’re wandering aimlessly for a contradiction. We’ll only see these proofs for certain question types in MAT137, but you’ll see more if you take PHL245. |
| eg. Prove *P* |
| Proof by Contradiction, suppose ~*P*  *Use ~P to arrive at a contradiction somewhere*  Thus, *P* |

## 2.4. Proof by Induction

Induction works for integers/natural numbers. While not common in MAT137, it’s valuable in CSC111/ CSC148/CSC165 due to ties to recursion and growth rates (Big-O).

Say you want to prove . You notice vs. Clearly, never catches up to . But just because it *seems* right doesn’t mean it’s a proper mathematical proof! Do it like so:

|  |  |
| --- | --- |
| **Induction** | is a statement with the variable , like .  refers to the statement when , like .  Statements with variables are used studied in **predicate logic**, which is more important for computer science than MAT137.  Remember to add “Let …”, since the statement starts with .  The logic behind this is: since is true, is true as for . Since is true, is true, and so on. |
| eg. Prove |
| Proof by Induction, let  Base Case:  *Find a way to show*  Inductive Step:  Assume  *Find a way to show* |

# 3. Limits

## 3.1. Theory

### 3.1.1. Definition ()

Any value of has a corresponding distance to , .

Any value of has a corresponding distance to , , which we’ll call .

What does mean?

The closer gets to (while not touching it), the closer gets to .

As the distance between and becomes infinitesimally small (but not 0),

the distance between and becomes infinitesimally small (and maybe 0).

If you manually picked a that’s close to (where ), I could always pick a closer to than what you just picked (where ).

eg. , and thus

* If you pick , I pick , so , which is closer to .
* If you pick , I pick , so , which is closer to .
* If you pick , I pick , so , which is closer to .

Let’s gradually turn the above into a logical statement.

1. No matter what you pick, I can always pick a so that is closer to than .
2. For any distance between and you pick, there is always a such that the distance between and is smaller than the distance between and .
3. For any , there exists such that is smaller than .

Why is ? Because we defined as a distance (an absolute value); it cannot be negative. It cannot be 0 either, because we’ve defined the limit as when but isn’t 0. So .

But this isn’t the full definition of limit. Instead of saying there exists , where , it’s better to say there’s a **range** of values that give closer values to than what you choose.

eg. , and thus

* If you pick , I can pick any where for a closer .
* If you pick , I can pick any where for a closer .
* If you pick , I can pick any where for a closer .

Let’s gradually turn the above into a logical statement.

1. No matter what you pick, I can always pick in the range (where ) that’ll give me a closer to than .
2. For any distance between and you pick, there’s always a where any in the range will make smaller than .
3. For any , there exists such that if , is smaller than .

How did I get?

|  |  |
| --- | --- |
|  | Since ,  To make the inequality work, we set . If you’re confused, see 3.1.2. |

### 3.1.2. Notation

|  |  |
| --- | --- |
| Consider the definition of limit of : |  |
| Note that it uses but never introduces it. The statement should really be: |  |
| You could also write it like: |  |
| Don’t use the 2nd form; it doesn’t work in negations. Takeaway: limit should work for *all numbers* (ie. ) | |
| We could actually further expand this, but at the cost of getting even bulkier. |  |

So, beware of these notation shortcuts in MAT137 and future math courses:

* If there’s an un-introduced or , assume it’s and
* If there’s a variable like with specified no number set (eg. ), assume it’s in

By saying , we implicitly mean exists.

is a placeholder for , which is clunky, but you may need to write that, so get familiar with:

### 3.1.3. Absolute Value

Limit proofs heavily involve absolute value, so make sure you’re familiar with how they work.

|  |  |  |  |
| --- | --- | --- | --- |
|  | |  |  | | --- | --- | | *(“Multiplicative”)*  *(“Non-Degenerate”)* |  | |

### 3.1.4. Types of Limits

One-Sided Limits

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Consider .   |  |  |  | | --- | --- | --- | | does not exist (DNE). | (the limit from the right/+ side) | (the limit from the left/- side) | |  |

is *notational shorthand*. As a mathematical statement, it’s meaningless as we can’t compute with . Also, alternate notation:and .

Lastly, my prof saw one-sided limits as a lower-level math thing, but I don’t remember why.

Approaching Infinity

|  |  |
| --- | --- |
| Consider shown on the right.  (the limit going to the right)  (the limit going to the left)  In both cases, our limit definition breaks down & stops making sense:  So, instead of a value of defining a range of , we use a lower bound : |  |

The intuition is that I can always “choose” a bigger with an closer to than the you chose.

### 3.1.5. Proving a Limit

|  |  |
| --- | --- |
| eg. Prove |  |
| *(Let , let )*  Let  Pick …  Assume  Show  … | *Expand the definition of limit.*  *Set-up the proof by introducing necessary variables.*  *DON’T pick a value initially! You don’t know what it has to be yet, so leave it blank for now.*  *You can add “Let ” under “Pick …”. It’s optional. Note you can’t set , as you haven’t introduced yet when you introduced . It’s picky.* |
| *Rough Work*  *The goal is to get to .*  *Our assumption, , is key to helping us. Turn into something involving .*  *Since ,*  *If , then we proved our objective. Let’s try simplifying first, using our assumption.*  *Uh oh. We can’t smoothly turn into here. What if we picked a such that ?*  *Then . But wait, , so , but is arbitrary and positive, so is not always true?*  *Yes. So, let’s also pick such that Then we’d get*  *How can we pick and ? Remember a theorem from 2.2, . If we pick we get and , which works.* | *In the rough work, we work backwards from the solution (since we can’t do that in a normal proof)*  *Why? So we can use the assumption. You’ll see how.*  *Remember in a limit, Here, so we’re not actually dividing by 0 when we do this step.*  *As shown by 3.1.3’s refresher on absolute value*  *Since , multiplying it on both sides doesn’t flip the inequality.*  *Some might pick , resulting in*    *We cannot do this. Why? Recall the limit definition: Since we introduced after , we cannot base on .*  *Remember we can pick any . Choosing is arbitrary; the proof works with other choices. But is simple and almost always works, so I’d recommend it as the default. We’ll see when it doesn’t work.*  *I picked to turn into the elegant, readable . Any smaller number works, but isn’t as pretty.*  *There’re many concepts here that might make you think “How was I supposed to think of that?” The answer is…experience. But don’t worry – all other “prove a limit” questions will have a similar format.* |
| Let  Pick , then and *.*  Assume  Show  Since *,*  Since *,*  Since | *Return back to our formal proof.*  *Note that is just one solution pair. If you don’t pick , you can still solve the question, but you might have to change to a different .*  *Essentially just copy your rough work in.*  *To summarize, our chain of logic in this proof is:*   |  |  | | --- | --- | |  |  |   *Odds are, most limit proofs will follow this logic.* |

I never proved one-sided limits in MAT137, so I doubt it’ll be a thing on the exam.

### 3.1.6. Proving a Limit ()

|  |  |
| --- | --- |
| eg. Prove |  |
| *(Let , let )*  Let  Pick …  Assume  Show  … | *Expand the definition of limit. I choose to re-write it, simplified here, for clarity’s sake, but it’s not strictly necessary; you can “derive” it in the last steps of your proof anyways.*  *As usual, leave the variable you’re picking blank and do some rough work first.* |
| *Rough Work*  *We need .*  *If and , then we can achieve that.*   |  |  | | --- | --- | |  |  |   *There are two solutions: or .*  *Repeat this above calculation with instead to get and .*  *Remember that x > 0 and we choose N where x N, so we have to choose or . The former has a stricter bound, so we’ll set N based on that.* | *Splitting the inequality into two equations*  *Why choose the stricter bound? Because if , then by extension, . We get more tools to work with this way.* |
| Let  Pick  Assume  Show  Since ,            Since and ,      Thus    From the triangle inequality, | *Make sure to check that since , and thus satisfies . This is another reason why out of the 4 possible solutions, we chose – it’s the only one guaranteed to be positive!*  *We can divide by as we know , thus . By the same reason, applying absolute value changes nothing since .*  *Recall the theorems section in 2.2. The triangle inequality is often used in limit proofs because of the absolute values involved.*  *This proof type is less common than 3.1.5.* |

To prove , do

To prove , do

I’ve never seen these proof types in my experience of MAT137, so don’t worry about it too much.

### 3.1.7. Disproving a Limit

|  |  |
| --- | --- |
| eg. Prove |  |
| Pick …  Let  Consider …  Show  Show  … | *Negate the definition and prove it true.*  *I use “Consider”, not “Pick” since it’s a counterexample, but I think both work. I’m fuzzy on the notation myself; just do what your math professor does.* |
| *Rough Work*  *To satisfy , we can set something like .*  *To satisfy , let’s derive it from and .*  *Since ,*          *So we need to set if we want .* | *We can base on because we introduced first.*  *Since we picked a specific , we can calculate a specific .*  *As ,*  *As*  *You need to arrive at this realization to come up with the inequality* |
| Pick  Let  Consider  Show    Since ,  Show  Since , | *Any value of where works.* |

## 3.2. Computation

### 3.2.1. General Methods

When computing , simplify first, then substitute .

|  |
| --- |
| eg. Evaluate . |
|  |

* You can **cancel out** the because when , .

|  |
| --- |
| eg. Evaluate . |
|  |

* *Polynomials*, limit approaches divide by the **polynomial with the highest exponent**
* Note that lower powers don’t matter:.

|  |
| --- |
| eg. Evaluate . |
|  |

* *Exponents*, limit approaches divide by the **exponent with the highest base**.

|  |
| --- |
| eg. Evaluate . |
|  |

* *Exponents*, limit approaches divide by the **exponent with the lowest base**.

|  |
| --- |
| eg. Evaluate . |
| |  |  | | --- | --- | |  |  | |

* *Absolute values* split into two **one-sided limits**.

|  |
| --- |
| eg. Evaluate . |
| |  |  |  | | --- | --- | --- | |  |  |  | |

* *Special case*, approaches split into two **one-sided limits**.

|  |
| --- |
| eg. Evaluate . |
|  |

* *Square roots* use the relation , **multiplying by conjugate**, to simplify

|  |
| --- |
| eg. Evaluate . |
|  |

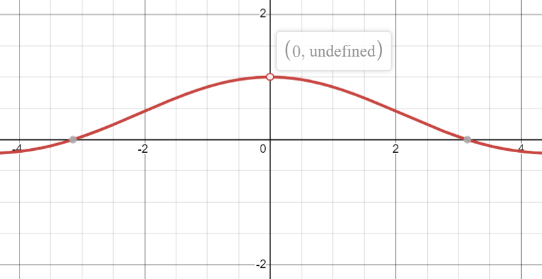
* compute the upper limit first, then the exponent.
* *Note:* In math, it’s convention that . In science/engineering, and usually . In computer science, sometimes . I use “” because I like it.

|  |
| --- |
| eg. Let . Evaluate . |
| |  |  |  | | --- | --- | --- | |  |  |  | |

* *Piecewise functions* split into two one-sided limits based on the piecewise function.

In higher-level analysis, **gamma functions** are used for limit/derivative of factorials (eg. ). Also, it’s bad practice to write or ( is mathematically improper). I only write it to be explicitly clear.

### 3.2.2. Limit Techniques

Small Angle Approximation

is on the right. You will learn to prove some of these “dodgy” theorems in the Taylor Series section.

|  |
| --- |
| eg. Evaluate . |
|  |

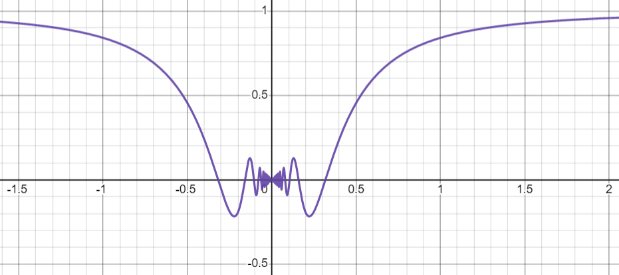
* *Sine* divide by **whatever’s inside the sine**, use small angle approximation

|  |
| --- |
| eg. Evaluate . |
|  |

* *Tangent* **write in terms of sine and cosine**, and use small angle approximation
* Note that , but . Both numerator and denominator must approach 0

Squeeze Theorem

|  |  |  |
| --- | --- | --- |
| * at point , and |  |  |

Most likely, you’ll be using squeeze theorem for , where you must show .

|  |  |
| --- | --- |
| eg. |  |
| Since , by the squeeze theorem, .  *Another way (may be easier, as you only calculate a limit on one side):*  Since , *(Absolute Value properties, see 3.1.3)*  By the squeeze theorem, , and thus |  |
| |  | | --- | | *Squeeze Theorem Tools:* | |  |   *You’ll formally learn these in 6.1.2.* |

# 4. Derivatives

## 4.1. Theory

### 4.1.1. Definition as Limit

For all , the derivative at any is:

Remember this formula! Sections 4.1.2 and 4.1.3 explain this formula’s applications, but aren’t important.

can be further differentiated to and and so forth.

The *n*-th order derivative, , is the number of times is differentiated.

and is Leibniznotation, equivalent to (where you’re differentiating ).is *.*

### 4.1.2. Definition as Slope

Consider any two points and , which are on the graph of . Take their slope:

And what happens as becomes infinitesimally small? You get the same formula as 4.1.1.

### 4.1.3. Definition in Kinematics

Derivative can be understood as “rate of change” in physics.

**Distance/Position (d):** The net distance an object travels.

**Displacement (s):** The net distance an object travels, relative to the origin.

**Velocity (v):** The *rate of change* of an object’s displacement/position/position.

**Acceleration (a):** The *rate of change* of an object’s velocity.

Let be an object’s position at time . The object’s average velocity between points of time and is

And what happens as becomes infinitesimally small? You get the same formula as 4.1.1.

### 4.1.4. Derivative Properties

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **General Theorems** |  |  |  |  |
| *Chain Rule:* |  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

For “Prove …” question with a derivative, plug and compute a solution. There is no actual “proving” like with limits. I never did derivative proofs in class, nor were there these proofs on the exam. When computing derivatives, don’t use the 4.1.1 formula as it’s clunky and awkward. Use the above table.

|  |  |
| --- | --- |
| eg. Prove. *(From MAT137, April exam 2018)* | *This is probably the extent of the derivative proofs you might get on the MAT137 exam.* |
| Let .   |  |  | | --- | --- | |  |  | |

|  |  |
| --- | --- |
| eg. Prove. | *I don’t recommend practicing derivative proofs with theorems. Just know that this is how to approach it.*  *This step’s tricky to figure out.* |
| Let . |

## 4.2. Computation

### 4.2.1. General Methods

Use the table from 4.1.4 for most, if not all computations. Here are some examples.

|  |  |
| --- | --- |
| eg. Evaluate . | *Apply*  *Apply*  *Apply* |
|  |

|  |  |
| --- | --- |
| eg. Evaluate . | *Apply*  *Apply*  *Apply*  *Apply* |
|  |

|  |  |
| --- | --- |
| eg. Let. Find . | *Apply*  *Apply*  *Apply*  *Apply* |
|  |
| DNE at as *(one-sided limits)* | |

### 4.2.2. L’Hôpital’s Rule

This is a special case for computing more complex limits:

When , apply this rule.

When , apply this rule.

When ,

* Re-write if is simpler than , then apply this rule.
* Re-write if is simpler than , then apply this rule.

|  |  |
| --- | --- |
| eg. Find . |  |
| *Apply L’Hôpital’s Rule,*              *Apply L’Hôpital’s Rule (again),* | *You have , where and . Since is simpler, set it to the denominator. This is for simplicity in calculations.*  *As ,*  *Some questions will make you have to apply L’Hôpital’s rule many times.* |

L’Hôpital’s Rule has some preconditions, which *usually* aren’t a problem:

* or , called **indeterminate form**.
* and exist, and the denominators aren’t 0
* exists or is equal to

Its complete proof is way too long-winded for MAT137. Many profs look down on this rule as it’s “overused”, like a “magic wand” to mindlessly plug numbers without understanding important concepts.

### 4.2.3. Logarithmic Differentiation

|  |  |  |
| --- | --- | --- |
| For a product/quotient of functions or exponent functions, like  take ln of both sides and differentiate. | *Refresher on Logarithm Operations* | |
|  |  |

|  |  |
| --- | --- |
| eg. Let . Find . |  |
|  | *You could try doing quotient rule or chain rule instead, but it won’t be fun.*  *Using chain rule,*  *I’m not expanding this because it’s difficult and not enlightening.* |

These questions are cruel because it’s just mindless and time-crunching. I doubt they’ll be on the exam, but just know that the method exists.

### 4.2.4. Implicit Differentiation

When differentiating an equation where both variables cannot be isolated, treat each variable as a function when taking derivatives.

|  |  |
| --- | --- |
| eg. Find and of at . |  |
| Substitute into both equations: | *You can simplify x’, but not y’, so keep y’.*  *It’s better to write , not , as the question writes it that way. I use y’ to save space.*  *Replace y’ with what you found it to be.*  *Implicit differentiation can get nasty past the first-order derivative.* |

You will never find 3+ variables (MAT137 is one-dimensional calculus). I believe implicit differentiation is also not common in higher-level math. Logarithmic differentiation is technically implicit differentiation.

## 4.3. Application

### 4.3.1. Optimization

There’s a variable that needs to be minimized/maximized, .

There’s a variable that can changed, .

Calculate , and solve for .

|  |  |
| --- | --- |
| eg. Find the farthest point(s) from on the ellipse . | *(From MAT157, April exam 2018)*  *From Pythagorean theorem*  *Isolate x (you can choose to instead isolate y, but plugging it into the distance equation will be more annoying)*  *Always simplify until your equation has only 1 variable left in it.*  *The variable to maximize is D*  *The variable that can change is y.*  *Do and solve for y.*  *Plug y in the distance formula to find x.* |
| The distance between and is  We know must be on the ellipse .  Substitute this into the distance equation:  and are the farthest points from . |

|  |  |
| --- | --- |
| eg. You want to sell a square open box of volume 1,000 cm3 for $1,250: the bottom is a square, the four sides are identical rectangles, and the top is empty. The material for the bottom costs $2/cm2 and the material for the four sides costs $3/cm2. Is this operation profitable? | *Since base = width (it’s a square)*  *The top is empty. It’s bw, not 2bw.*  *For these questions, you will often have to figure out the formulas yourself, which can be hard.*  *The variable to minimize is C*  *The variable that can change is w*  *Do and solve for y.*  *This is unsolvable without a calculator. Good news: the exam doesn’t allow calculators, so you’ll get something easy to compute.* |
| We have a square prism of volume 1000.  We know the total surface area is:  The total cost is based on surface area, and is thus:  Since the cost of a box is exceeded by the profit , this operation is barely profitable. |

### 4.3.2. Related Rates

Write an equation relating all variables.

Perform implicit differentiation against variables that change with time (*dt)*, and plug the right values in.

|  |  |
| --- | --- |
| eg. The volume of a cone is increasing at 30 m3/min. Its diameter and height are always equal. How fast is the cone’s height increasing when the height is 10 m? | *It’s a good idea to write all information given in the problem and what to solve. Kind of like physics questions.*  *You can replace h with 2r instead, but remember, the question asks for , not*  *Substitute the initial values*  *Isolate* |
| We know the volume of a cone is  The cone’s height is increasing at a rate of m/min. |

Related rates is closely tied to implicit differentiation, both concepts originating from physics.

# 5. Continuity

## 5.1. Theory

### 5.1.1. Definition

All standard functions (ie. ) are continuous in their domains.

Adding/subtracting/multiplying/dividing functions retains their continuity (except division by 0)

You might see “*Let be a function defined at least around on an interval centered around , except maybe at .*” It means is continuous around , but not necessarily at – the bare minimum for to exist.

### 5.1.2. Types of Discontinuities

|  |  |
| --- | --- |
| Removable Discontinuity   * exists * or DNE | Jump Discontinuity   * , so DNE |
| eg. DNE at  “Removable” as you can redefine as a piecewise function & erase the discontinuity | eg. |
| Infinite Discontinuity   * or | Essential Discontinuity  Anything that doesn’t fit into the other types. |
| eg. | eg.  Is described as “oscillating wildly” as it approaches 0 |

## 5.2. Limit Laws

Let and . Then

|  |  |
| --- | --- |
| * *(additive)* | * *(multiplicative)* * (if ) |

* is continuous (*function composition*)

You’ve been implicitly using these in your computations; now we’ve justified them rigorously. Also, the proofs for all these theorems, except multiplying/dividing, are easy enough to be learned in MAT137.

|  |  |
| --- | --- |
| eg. Prove |  |
| Assume , which means  Show , which means  Let  Pick …  Let  Assume  Show | *Remember the premise of this theorem.*  *If your proof involves assuming another limit already exists, you’ll need to add the usually unnecessary part. Add numerals to disambiguate symbols.* |
| *Rough Work:*  *Note that can be simplified to*  *You already know*  *You already know*  *You want to show*  *So it’s just a matter of picking variables to match and modus ponens.*  *But there’re some things to watch out for. Consider your assumption ,*   1. *For , since it’s , you can pick anything and what follows will be true. So “Pick ” is fine.* 2. *For , you may not do “Pick …”, as you know , but not what is. Thus, you must do “Pick … = ”. Since we need and we can do “Pick …”, do instead.* 3. *For , the same principle as applies. is fine.* | *Divide both sides by*  *Modus ponens was in 2.2. Here’s a refresher of it:*  *You write “Pick …”, not “Let …” despite it being , because this is an already-known assumption.*  *You’re instantiating ; that is, since the assumption is true for all , it is thus true for this specific value of .”* |
| Assume , which means  Show , which means  Let  Pick  Let  Assume  Show  Pick , , then  Since , |  |

|  |  |
| --- | --- |
| eg. Prove |  |
| Assume , which means  Assume , which means  Show , which means  Let  Pick …  Let  Assume  Show | *Remember there are two premises for this theorem.*  *Theorems have complicated set-ups but much less rough work.* |
| *Rough Work:*  *Note that*  *You already know*  *You already know*  *You already know*  *You want to show*  *If you add and , you get*  *So we need to choose and where*  *Similar logic from the previous proof applies. and .*  *But wait. We need to pick AND pick . We can’t pick …*  *or …, so what do we do? The answer is .* | *But wait!*  *is different from what we show,*  *Luckily, we already have a tool for this: triangle inequality!* |
| Assume , which means  Assume , which means  Show , which means  Let  Pick , then and  Let  Assume  Show  Pick , , then  Pick , , then  Since ,  Since ,    From the triangle inequality, | *Hopefully with this, you now have a solid grasp of when to use min/max and the triangle inequality.* |

|  |  |
| --- | --- |
| eg. Prove is continuous |  |
| Assume exists, meaning  Assume is continuous  Since is continuous, it is continuous at  Since is continuous at ,    Show , meaning  Let  Pick  Let  Assume  Show | *From the premise. I’m not using placeholder .*  *Just try to follow this proof for now. I’ll explain the logic behind all my choices below.*  *I boxed all of the big statements for clarity.* |
| Pick , then  Since ,  *Case 1.*  Then *,* and  Pick , then  Since ,  *Case 2.* | *Modus ponens*  *Modus ponens* |

This theorem is the “Continuity Law for Composition”. **Composition** is when you take a function of another function, like , also notated . This proof is arguably the hardest one of MAT137.

*So, what just happened? How are we supposed to come to a conclusion like this on our own?*

|  |  |
| --- | --- |
| *Let’s start with* ***what we know****:*   * *is continuous* | *And what we* ***need to show****:* |

*With our first two assumptions, we can get .*

*But our conclusion’s structure is different: vs. .*

*Remember*  *is continuous, . Let’s expand that:*

*Pick , then*

*Compare and the conclusion, . This statement is the missing link! Our chain of logic for this proof is thus summarized below:*

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *continuous at* |  |  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |

*Wait a minute, we only have . We need for the conditional!*

*That’s why we split into two cases:*

* *, thus and we can finish the proof as shown above.*
* *, where we have to solve the proof in the other way, as shown in the proof.*

*Note that this detail, while mathematically rigorous, is technical and omittable (according to my MAT137 prof)*

|  |
| --- |
| eg. Explain why the following methodology is incorrect *(Note: Not an exam question)*. |
| The first step uses the limit law .  This law presupposes both and exists. However, does not exist. |

## 5.3. Differentiability and Continuity

|  |  |
| --- | --- |
| eg. Give a function that is continuous but not differentiable at . | *We combined one-sided limits and the definition of continuity here.*  *Note that is also a valid solution. Try* |
| Take .  Or you prove, , style, that . Moving onto derivatives,     |  |  |  | | --- | --- | --- | |  |  | The one-sided limits aren’t equal, thus DNE at 0 and isn’t continuous at 0. Thus isn’t differentiable at . | |

## 5.4. Important Theorems

These theorems seem obvious intuitively, but their formal proofs are actually too hard for MAT137.

### Intermediate Value Theorem | Brilliant Math & Science Wiki5.4.1. Intermediate Value Theorem (IVT)

If is continuous from to , there’s a where

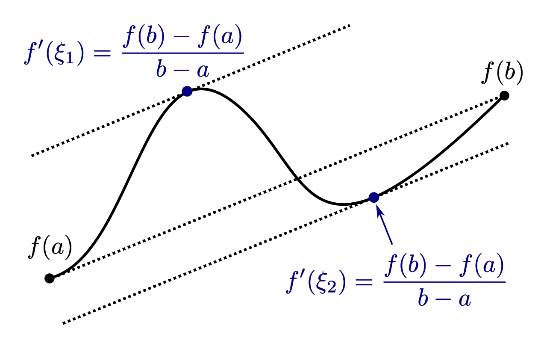
*“If and , must pass somewhere from to .”*

### Extreme Value Theorem Visualization – GeoGebra5.4.2. Extreme Value Theorem (EVT)

If is continuous on closed interval , there’s a minimum/maximum in that interval.

If necessary, you can expand the definition of maximum as such:

### 5.4.3. Mean Value Theorem (MVT)

If is continuous between and , there’s a with the same slope as the average slope between and .

*“If you accelerate from 80 km/h to 100 km/h, you reach 90 km/h sometime in between.”*

My prof called this the Most Valuable Theorem (MVT). I think it’s because it relates derivatives to continuity. If you’re wondering why the boundaries , differ…I don’t know.

### 5.4.4. Proofs with Important Theorems

These theorems tell us , but not how to find , so proofs with them are often more abstract. It’s a good idea to remember these theorems, because proofs with them are not uncommon on exams.

|  |  |
| --- | --- |
| eg. Show there are at least 3 solutions to . |  |
| Let   |  |  | | --- | --- | |  |  | |  |  |   Since is continuous, by the Intermediate Value Theorem (IVT), | *Unfortunately, you have to just guess and check to make sure you have the right values.*  *f(x) is continuous since it is composed of standard functions, and adding standard functions retains continuity.* |

|  |  |
| --- | --- |
| eg. Let be differentiable. If has 1 zero on [, how many zeroes does have at most on [? | *MVT is a rare instance where you must use proof by contradiction.*  *In this case, the opposite of*  *“at most x” is*  *“at least x + 1”.*  *You can learn about symbolizing English sentences in PHL245.*  *Note that* |
| *ANSWER:* has at most 2 zeroes.  Proof by contradiction, suppose has at least 3 zeroes on :  where .  Since is differentiable, it is continuous. By the Mean Value Theorem,  Thus and has 2 zeros on . This contradicts the premise that has 1 zero on .  Therefore, has at most 2 zeroes on . |

This specific usage of MVT is called **Rolle’s Theorem:**

It’s derived from the MVT (ie. a **corollary**). I guess it’s included because it’s a common usage of MVT.

|  |  |
| --- | --- |
| eg. Let . Suppose is differentiable on and . Show is increasing on . | *(From MAT137, August exam 2019)*  *MVT, so proof by contradiction!*  *Technically, the MVT requires*   *being continuous on , but it’s a technical detail. To be rigorous:*  *Pick such that and . Then you can get is continuous on and differentiable on to apply MVT.* |
| Let  Assume is differentiable on  Assume  Show is increasing on , meaning  Let  Assume  Show  Proof by contradiction, suppose  Since is differentiable on , by the MVT,    Since ,  Since ,  Then  This contradicts assumption |

# 6. Inverse Functions

## 6.1. Theory

### 6.1.1. Definition

|  |  |
| --- | --- |
| Let be a function.  The **domain** is set *A* (set of all inputs of).  The **codomain** is set *B*.  The **image** is the set of all outputs of . | *Don’t use the word* ***range*** *as it’s ambiguous: depending on the mathematician, it can mean codomain or image.* |

|  |  |
| --- | --- |
|  | “ *can reach any value in the codomain*.”  “*Each* *y* *has exactly one* *corresponding* *x*.” |

“ is invertible” means there exists an inverse function where .

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| The **Inverse Function Theorem** finds the inverse’s derivative. Its generalized proof is very difficult. | |  |  |  |  |  | | --- | --- | --- | --- | --- | |  |  |  |  |  | |  |  |  |  |  |   *Important:* is defined as the inverse of .  Visually, looks like reflected across . |

|  |
| --- |
| *Consider where .*  is not onto, as . Restrict **codomain** to make **onto**.  is not one-to-one, as . Restrict the **domain** to make **one-to-one**. |

### 6.1.2. Inverse Trigonometric Functions

|  |  |  |
| --- | --- | --- |
| **Function** | | |
|  |  |  |
|  |  |  |
| **Inverse Function** | | |
|  |  |  |
| The inverse of with the domain cut to to make it one-to-one and onto | The inverse of with the domain cut to to make it one-to-one and onto | The inverse of with the domain cut to to make it one-to-one and onto |

Why the domains are cut like that is completely arbitrary – the math community just decided it was convenient this way. Note that , , and exist, but it’s uncommon (not in MAT137)

Trigonometric functions on their inverses have interesting graphs. You’ll learn how to calculate these:

|  |  |
| --- | --- |
| **Triangle Wave** | **Sawtooth Wave** |
|  |  |
|  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

|  |  |
| --- | --- |
| **Square Wave** | Waves aren’t in MAT137, but I think there’re in MAT237’s Fourier Series. Waves are used in electrical & computer engineering for electronics & signal processing, as well as in creating electronic music through additive synthesis. |
|  |
|  |

## 6.2. Computation

Finding

Write the equation in terms of *y* and *x*.

Write the equation again, replacing *y*’swith *x’s* and vice versa.

Isolate *y*, and rewrite *y* as .

|  |  |
| --- | --- |
| eg. Find the inverse of . | *Remember to swap the x and y inside domains as well!* |
| |  |  | | --- | --- | |  | *Write the domain in terms of x.* | |

|  |  |
| --- | --- |
| eg. Find the inverse of . | *The technique you use is called* ***completing the square:*** |
| |  |  | | --- | --- | |  | *There’re 2 solutions as isn’t onto* | |

|  |
| --- |
| eg. Find the inverse of . |
| |  |  |  | | --- | --- | --- | | *Replace with for clarity.*  *Complete the square (hard!)* | *See how this can be cleanly square-rooted.* |  | |

|  |  |
| --- | --- |
| eg. Evaluate with logarithmic differentiation and with . | *Choose the method you’re most comfortable with.*  *The relationship is* ***essential*** *to solving limits and derivatives of this form .* |
| |  |  | | --- | --- | | *Method 1.* | *Method 2.* | |

|  |  |
| --- | --- |
| eg. Prove (*Hint*: First show ) | *In all of MAT137, I never saw log with a base that wasn’t e. Nor was this in any exams. So this formula isn’t important.* |
| |  |  | | --- | --- | | *From the log rules table in 4.2.3,* |  | |

|  |  |
| --- | --- |
| eg. Let be one-to-one. Find and | *(From MAT137, April exam 2019)*  *This question is very gimmicky.*  *You have to plug in values of until you guess a working one. Usually, the question is nice and gives you a simple number.* |
| *Finding the way I showed you won’t work.*  *Good luck trying to isolate y; that is, you can’t. Instead, realise this:*  *To calculate , you substitute into .*  *Thus, to calculate , you substitute into .*  *When ,*  *Thus We know there’re no other solutions as is one-to-one. From the inverse function theorem,*      *Differentiate to get , then* |

Trigonometric Inverses

Fit in the right domain using the following rules:

|  |  |  |
| --- | --- | --- |
| *Refresher on Trigonometric Identities, Part 1* | | |
|  |  |  |

|  |  |
| --- | --- |
| eg. Evaluate . | *Remember the restricted domain for sine and arcsine:* |
| *is not inside , so we change it so it is.* |

|  |  |
| --- | --- |
| eg. Evaluate . | *Use*  *Use*  *Use*  *Since ,* |
|  |

|  |  |
| --- | --- |
| eg. Evaluate . | *Remember tan and arctan’s domain:*      *Thus*  *Use Pythagorean Theorem to calculate the hypotenuse.* |
| , where  *This means . We can calculate*  *from this triangle ratio.* |

# 7. Graphing

## 7.1. Simplifying Rational Fractions

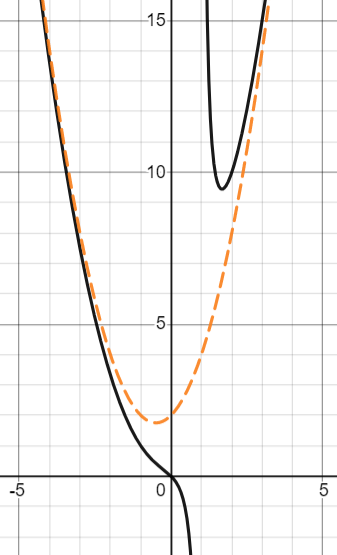
**Polynomial:** A function of format

**Degree:** A polynomial’s highest power (eg. for the example in the line above, )

**Rational Function:** A function of format , where and are polynomials.

Concepts from 7.1 will always be used in tandem with other concepts on the exam, never on their own.

### 7.1.1. Long Division

****Do this for when the **degree** of the **degree** of .

|  |  |
| --- | --- |
| eg. Simplify with long division. | |
|  | *Not much to say other than…just do long division, but instead of base 10, it’s base x.*  *Note: The non-fraction part (orange) is the slant asymptote (see 7.4) if it’s a line.* |

### 7.1.2. Partial Fractions

Do this for when the **degree** of the **degree** of .

|  |
| --- |
| eg. Simplify with partial fractions. |
| *Note the opposing , , , on both sides. Set up a system of equations to solve them:* |

How did I know how to choose the numerator?

* Since , we add the form in the numerator.
* Since , we add the form in the numerator.

|  |
| --- |
| eg. Expand into partial fractions. |
| *A, B, C, D, E, F, H are all constants that one would need to solve for.* |

The proof for why partial fractions work is too difficult for MAT137, so just have faith in it.

## 7.2. Basic Properties

**Domain:** All possible inputs.

* eg. The domain of is , also writable as and

**Image:** All possible outputs.

**Zeroes:** Number of “solutions” or “instances when ” or “*x*-intercepts”

**Y-Intercept:** The value of when

**X-Intercept(s):** The value(s) of when

**Odd:** A function where

**Even:** A function where

**Discontinuities:** Areas where DNE. Review 5.1.2, types of continuities, for how to calculate them.

**Tangent Line:** A tangent line for at is a line touching ) with slope .

|  |  |
| --- | --- |
| eg. Let with domain . Identify all points of discontinuity, and their type. | *(From MAT157, April exam 2018)*  *We can ignore here because , so will just be or .*  *We have to make sure the numerator being 0 doesn’t change the infinite discontinuity.*  *Apply L’Hôpital’s Rule and realize the limit still diverges.*  *We can ignore here because when , is neither 0 nor* |
| is discontinuous when the denominator of is 0.  This is an **infinite discontinuity** (dividing by 0 gets infinity). But the numerator, , is 0 at      So the discontinuity is still infinite. Moving onto , this is discontinuous when the denominator is 0, or .      This is a **jump discontinuity** (ie. “jumps” from – to +). |

|  |  |
| --- | --- |
| eg. Find the tangent line to at . | *I got something like this in my MAT137 exam. Realize these questions only work in extremely specific conditions: that is, y is isolated if you set x = 0.* |
| Thus the tangent line passes . Do implicit differentiation:  Thus is the tangent line. Substitute into it: |

## 7.3. Derivatives and Graphs

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *Increasing* | *Non-decreasing* | *Decreasing* | *Non-increasing* | *Constant function* | *One-to-one* |
|  |  |  |  |  |  |

|  |  |  |  |
| --- | --- | --- | --- |
| **Extrema** | | | |
| **Global Extrema** | | **Local Extrema** | |
| **Global Maximum** | **Global Minimum** | **Local Maximum** | **Local Minimum** |
| Maximum of | Minimum of | Maximum of relative to surrounding points | Minimum of relative to surrounding points |

|  |  |  |  |
| --- | --- | --- | --- |
| **Inflection Point** | | **Critical Point** | |
| When switches between concave up/down | | When or | |
| **Concave Up** | **Concave Down** | **Local Maximum** | **Local Minimum** |
| follows a “” shape  is increasing | follows a “”shape  is decreasing | to its left, to its right | to its left, to its right |
|  |  | and | and |

|  |  |
| --- | --- |
| Visually, notice that and have one inflection point at .  is concave down concave up  is concave up concave down |  |

|  |  |
| --- | --- |
| eg. Let . Find its maximum and minimum on . | *(From MAT157, April exam 2017)*  *Doing is a common tool you should learn to use.*  *This is a “common sense” thing. If is always growing (or 0), its max is the right-most point in the domain, and vice versa.* |
| is non-decreasing, therefore:  Its maximum is the right-most , where .  Its minimum is the left-most , where . |

|  |  |
| --- | --- |
| eg. Let be continuous. Consider the graph of on the right. Describe at the points . | *(From MAT137, April exam 2019)*  *By “remains negative/positive”, I mean g’(x) is negative/positive to the left & right of g’(x)*  *By “changes*  to *”, I mean left of g’(x) < 0, and right of g’(x) > 0* |
| DNE, so it’s a **critical point**, but remains negative (no local extrema)  , so it’s a **critical point**; its sign changes to , so ’s a **local minimum**. is increasing around (not inflection point)  changes to , so it’s an **inflection point**. remains positive (not critical point)  , so it’s a **critical point**, butremains positive (no local extrema). changes sign to , so it’s an **inflection point**.  , so it’s a **critical point;** its signchanges to , so ’s a **local maximum**. is decreasing around (not inflection point)  DNE, so it’s a **critical point**; its sign changes to , so ’s a **local minimum**. is decreasing around (not inflection point) |

|  |  |
| --- | --- |
| eg. Find all inflection/critical points and graph | *Critical points are black*  *Inflection points are blue* |
| |  |  |  |  |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | |  |  |  |  |  |  |  |  |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |  |  |   *and follow concave up (), while and follow concave down ()*  *You might’ve done this in high school. Graph using the table as a guideline.* |

## 7.4. Asymptotes

Asymptotes are like tangent lines for as approaches infinity. may touch its asymptote(s).

|  |  |  |
| --- | --- | --- |
| Vertical Asymptotes  Vertical lines. When . Find when is undefined. | Horizontal Asymptotes  Horizontal lines. When . Find and . | Slant/Oblique Asymptotes  Slanted lines. Special case. Find with fraction long division. |
| eg.  Vertical asymptotes: | eg.  Horizontal asymptotes: | eg.  Slant asymptote: |

|  |  |
| --- | --- |
| eg. Give an example of a with vertical tangent lines at and , plus a horizontal asymptote at . | *(From MAT137, August exam 2019)*    *These questions should be very easy.* |
| A vertical tangent line at means . An easy way to do that is with .  We need a horizontal asymptote at . If we do , then and we get a horizontal asymptote. |

# 8. Integrals

## 8.1. Theory

### 8.1.1. Definition as Riemann Sum

What does mean?

The integral of at is the **area** between and the *x*-axis at .

* Note that if , the area is negative.
* **True area** refers to all area, negative or positive, while **net area** = positive area – negative area.

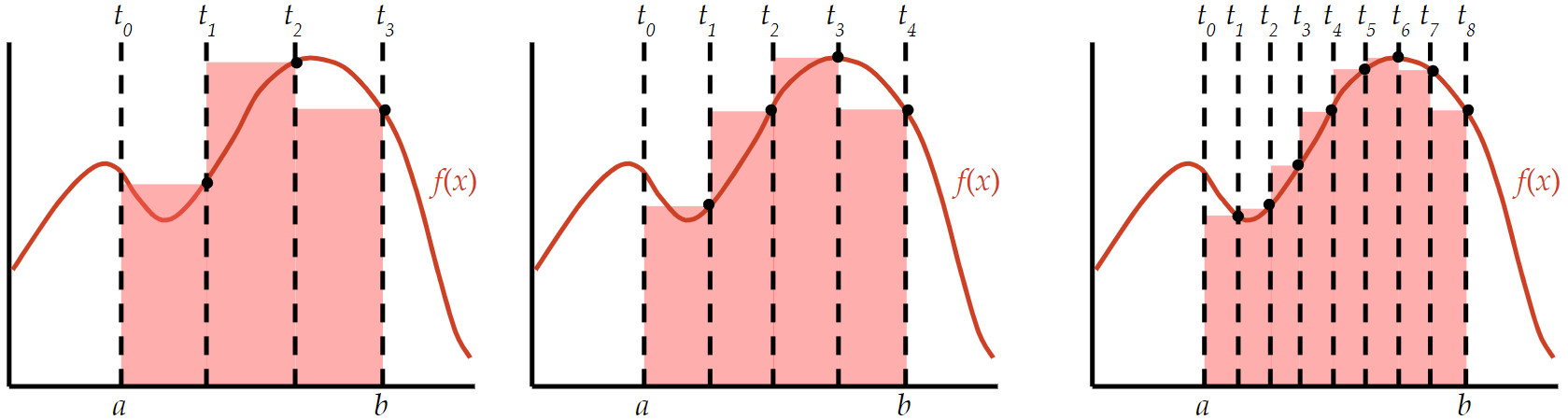
One way to calculate area is to *approximate* it by drawing rectangles under the function, and summing the area of those rectangles. Any sums calculated this way are called **Riemann sums**.

The **partition** is how we pick these rectangles. Specifically, it’s the values of shown below.

|  |  |  |
| --- | --- | --- |
| Let’s use 1 rectangle, its height defined as the **right-most** endpoint. Its area is  which is sucky and inaccurate.  Let’s use a partition of 2 evenly-spaced rectangles. The total area is |  |  |

Hopefully, you see the pattern. Given a partition of evenly-spaced points, , our area is equal to the sum of for all between 1 and *n*:

As the number of rectangles increases, the approximated area gradually approaches the net area.



Since all rectangles in the partition are evenly-spaced, we can simplify this formula to not use . I’m not going to explain this derivation because it’s neither hard nor important.

Then we add a limit to the formula: to calculate area of infinite rectangles.

We chose the **right-most endpoint** as each rectangle’s height as it gives the mathematically simplest formula.

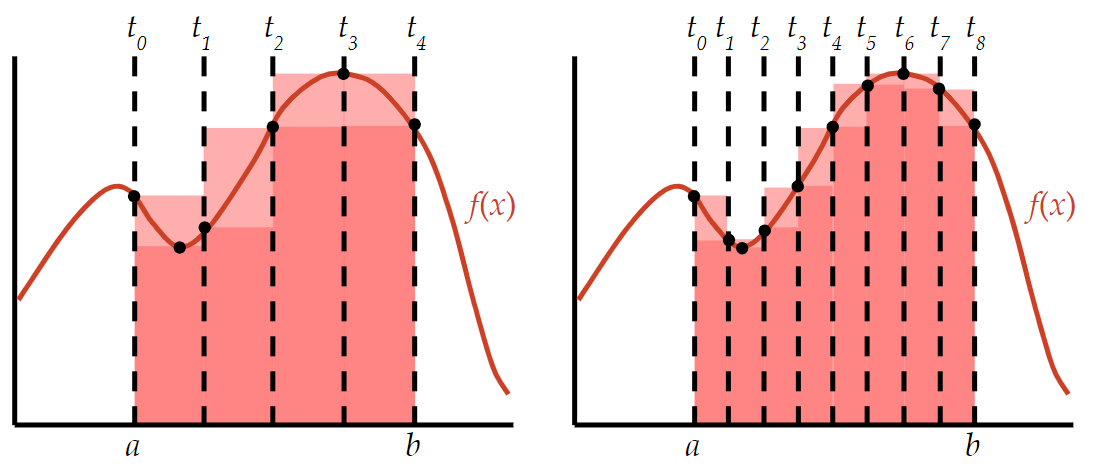
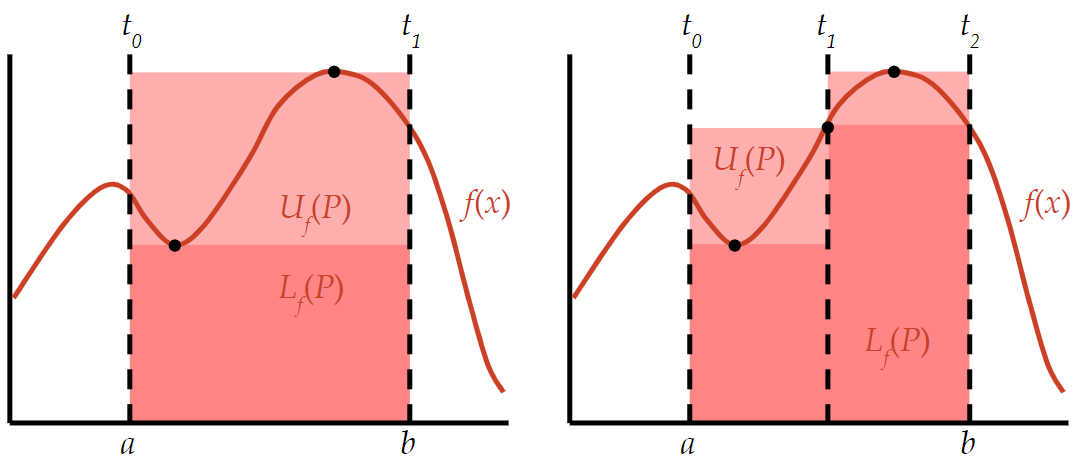
Calculating integrals via Riemann sums is unfortunately very difficult. In practice, we usually calculate integrals using the Fundamental Theorem of Calculus (8.1.6).

### 8.1.2. Integrability

While splitting up area into rectangles isn’t good for calculating, it’s good for proving integrability.

|  |  |
| --- | --- |
| Let’s use a 1-rectangle partition, *P.*  The height of , the **upper sum**, is ’s max value in *P*.  The height of , the **lower sum**, is ’s min value in *P*. |  |

Just like before, as we pick more rectangles in our partition, the approximation becomes more refined.



is integrable if and only if

The intuition is: matter what you pick, there’ll always be a partition accurate enough that the difference between and is smaller than . We must pick the partition ourselves.

Partitions are notated like “”, which translates to “a partition of 3 rectangles with bases going from to , to , and to ”.

### 8.1.3. Proving Integrability

|  |  |
| --- | --- |
| eg. Let . Prove if is integrable on . | *Where do values come from?*  *For the rectangle in ,*  *is*  *For ,*  *For ,*  *Repeat this for all six regions:*  *, , etc.* |
| is integrable:  Let  Pick , where  Show |

*The strategy here is to pick partitions to isolate regions with discontinuities, surrounding them with or .*

* *Our interval is and our discontinuities are .*
  + *For , we need a rectangle*
  + *For , we need a rectangle (I’ve shaded these areas in yellow.)*
  + *For , we need a rectangle*
* *Connect all points to finish our partition: “Pick ”.*
  + *Note the at the partition’s end. Our partition must cover all of .*

*In each isolated region, we know is true regardless of ’s value, since the rectangle in that region is defined as being centered on a discontinuity.*

* *For , and , even as approaches .*
* *For , and , even as approaches .*
* *For , and , even as approaches .*
* *Here, the rectangle always has a different height than the rectangle.*

*Between isolated regions, we know as there’re no discontinuities, so .*

* *For , and*
* *For , and*
* *and*
* *Here, the rectangle always has the same height as the rectangle.*

*Therefore, when you do and simplify, your answer should be . Pick to get .*

* *If you get the form , then you’ve chosen your partition wrongly.*

|  |  |
| --- | --- |
| eg. Let . Prove if is integrable on . | *Partition definitions can get wordy.*  *You could write this instead:*  *Pick*  *Since , therefore*  *But that’s just a technical detail and isn’t too important.*  *You get and need it to be . So you do “Pick n such that ”.* |
| is integrable:  Let  Pick , where , , and all intervals are equally-spaced, with width  Pick such that *(leave this blank until the last step)*  Show  Since is monotonic and decreasing, in any we know  and |

If is not monotonic, divide into sections that are monotonic and calculate area the same way.

|  |  |
| --- | --- |
| eg. Let . Prove if is integrable on . | *As the points become lower, they get denser. This graph has a gap as I manually inputted every point. I refuse to continue it past .*  *These are the most complicated partitions. The trick is to realise that even though there’re infinitely many “points”, they’re all approaching a cluster.*  *This summation notation isn’t really proper.*  *The good news is that no past MAT137 exam in the Old Exam Repository contains any very difficult questions like these.* |
| is integrable:  Let  There are finitely many where ,  Pick by constructing *n* rectangles around each with width  Show  Since there are infinitely many across  *Take the area of each individual rectangle that we constructed around each*   * *We know since we defined it that way* * *We know because visually,*   *Now take the area of rectangles not around each .*   * *We don’t know , but we know (we’re working in the interval , so the rectangle base widths add to that at most)* * *We know , because otherwise, any point where would be in .*   *Add these two values up to get* |

### 8.1.4. Disproving Integrability

|  |  |
| --- | --- |
| eg. Let . Prove if is integrable on . | *Why is this not integrable? Because there’re infinitely many “points” AND none of them are approaching any cluster.* |
| is not integrable:  Pick  Let be an arbitrary partition on  Show  Since there are infinitely many across  Since there are infinitely many across |

In conclusion, integrability has laxer conditions compared to differentiability:

### 8.1.5. Definition as Supremum/Infimum

Recall the relevant terms from section 1. Here are their definitions:

|  |  |  |
| --- | --- | --- |
| **Statement** | **Definition** | **English Translation** |
| *a* is an **upper bound** of *S*  *a* is a **lower bound** of *S*  *a* is the **supremum** of *S*  *a* is the **infimum** of *S* |  | Everything in *S* is smaller than *a*  Everything in *S* is larger than *a*  *a* is the smallest upper bound of *S*  *a* is the largest lower bound of *S* |

*Note*: “*S* is bounded above by *a*” means “*a* is an upper bound of *S”*.

We can alternatively define supremum and infimum as:

The intuition is that “anything smaller than will be inside the range of ”, and vice versa for infimum. Lastly, we can define the integral according to supremum and infimum:

### 8.1.6. Fundamental Theorem of Calculus (FTC)

My math prof emphasized this as the single most important takeaway of MAT137.

|  |  |
| --- | --- |
| FTC I  Let be continuous, . | FTC II  Let be continuous, . |

FTC II lets you to find **definite integrals** through finding the antiderivative of , .

**Indefinite integrals**, , represent a class of (the antiderivative) where .

* There’re infinitely many where .
  + eg. Let . Then
* In high school, it’s common to compute , where is the **constant of integration**. Higher-level math almost never uses because it’s misleading: it implies is a constant while in reality, is the set of all constants. Still, I believe MAT137 doesn’t care if you use .

If you did integrals in high school, you’ve probably been using the FTC unaware. You may have been led to believe that derivatives and integrals are “like opposites”. This is NOT TRUE. Math profs want you to learn that they are separate concepts, whose computations just happen to be related thanks to the FTC.

For definite integrals, we write instead of , since it’s compact. There’s also this notation, , which is trash as it’s ambiguous: is equal to or ?

### 8.1.7. Integral Properties

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **General Theorems** |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

We didn’t do any integral property proofs. They were in no past exams; I doubt they’ll be in a future one.

|  |  |
| --- | --- |
|  | **Mean Value Theorem (MVT) for Integrals:** |
| You can also define some functions as an integral:   |  |  | | --- | --- | |  |  | |

|  |  |
| --- | --- |
| eg. Let . Show that | *Make sure to swap the integral’s upper/lower bounds so that g(x) is the upper bound. Otherwise, the FTC doesn’t work.*  *Notation-wise,* ***it’s fine to omit “dx”*** *if it’s obvious x is being integrated. If you’re using a variable like t, add dt.* |
|  |

## 8.2. Computation

### 8.2.1. General Methods

U-Substitution

Set the equation’s most annoying part to (or anything else).

Do , isolate , and substitute it into the equation.

* Your equation must be written completely in terms of . It should be easier to integrate.

Integrate and replace with what you used it to substitute.

|  |  |
| --- | --- |
| eg. Evaluate | *Lesson 1: Set u to the annoying part of the equation. In a fraction, it’s usually the denominator.* |
| |  |  | | --- | --- | |  | The denominator is annoying, so we set that to . | |

|  |  |
| --- | --- |
| eg. Evaluate | *Lesson 2: Doing doesn’t always get rid of all the x, so you might have to do more.*  *Lesson 3: For definite integrals, note what the upper/lower bounds are written in terms of.* |
| |  |  | | --- | --- | |  | Set to the denominator, .  You CANNOT expand this yet. The upper/lower bounds are still defined in terms of , not . | |

You can define the bounds in terms of by substituting in : and . But it’s more work.

|  |  |
| --- | --- |
| eg. Evaluate | *Lesson 4: You can’t always set u to be the denominator. Choose strategically to cancel out the .* |
| |  |  | | --- | --- | |  | This is a special case.  *Simplify with* | |

|  |  |
| --- | --- |
| eg. Evaluate | *Note how limited this method is. What if the numerator is not ? Suffering.* |
| |  |  | | --- | --- | |  | Set denominator. | |

|  |  |
| --- | --- |
| eg. Evaluate | *Suffering.*  *First thing to realize is that if you set , then*  *, which fits nothing.*  *You can also try*      *As for the fraction with*      *You can try simplifying*      *into*      *and learn to compute in 8.2.3, which…includes completing the square. So you’ll have to learn to complete the square to simplify the denominator no matter what.*  *Lesson 5: You may have to u-substitute more than once.*  *And remember the formula for arctangent:* |
| *Set , then , and*   |  |  | | --- | --- | | *Set , then*  *and* |  | |

|  |  |
| --- | --- |
| eg. Evaluate | *Lesson 6: Be prepared to simplify with long division and partial fractions.*  *Use U-substitution to solve , set* |
| |  |  | | --- | --- | |  |  | |  | |

Integration by Parts

For an integral of format , set one to , the other to , then use these formulae:

|  |  |
| --- | --- |
|  |  |

I personally prefer internalizing it as this: Let be functions. Let . Then

|  |  |
| --- | --- |
|  |  |

Usually, you use this on .

* If , set .
* Otherwise, set

You may also use this on , where you actually treat it like , .

|  |  |
| --- | --- |
| eg. Evaluate | *We set and integrate by parts.*  *Remember .* |
|  |

|  |  |
| --- | --- |
| eg. Evaluate | *We have , where*  *So you do integration by parts. You will need to remember this gimmick.*  *You also do long division to arrive at this same step.* |
|  |

|  |  |
| --- | --- |
| eg. Evaluate | *Do integration by parts. You can treat either or as or here.*  *Add to both sides of the equation*  *This is another special case.* |
|  |

### 8.2.2. Trigonometric Integrals

|  |  |  |  |
| --- | --- | --- | --- |
| *Refresher on Trigonometric Identities, Part 2* | | | |
|  | *Divide by or to get the bottom two formulas* |  |

Here’s a guide of what to *u*-substitute in different circumstances:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | Reduce until there’s just one power of or . | |  |

Remember this integral:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  | Reduce until there’s a . | Reduce until there’s a . |

Cosecant/cotangent pretty much work the same way as secant/tangent.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  | Reduce until there’s a . | Reduce until there’s a . |

Note that not all integral cases are covered; it’s because those cases are too hard.

|  |  |
| --- | --- |
| eg. Evaluate | *These questions with sin/cos, sec/tan, and csc/cot, are fairly common on exams*  *I skipped u-substitution steps here to save space. In each case, I set* |
|  |

|  |  |
| --- | --- |
| eg. Evaluate | *I haven’t seen these question types on past exams, probably because it’s too based on remembering formulae.*  *Remember* |
|  |

### 8.2.3. Trigonometric Substitution

My math prof says there’s no practical point to this other than showing off that you can compute complicated integrals. Still, this might be on the exam, so be prepared.

|  |  |  |  |
| --- | --- | --- | --- |
| **Type 1.** | **Type 2.** | **Type 3.** | **Type 4.** |
| Set , then | Set , then | Set , then | Complete the square.        Then do 1/2/3 trig substitution. |

I suppose for type 1, you can also set instead. Do what you prefer.

|  |  |
| --- | --- |
| eg. Evaluate . | *We have type 3, , with*  *Differentiate on*  *Remember that sine, cosine, and tangents are defined as ratios of side lengths of triangles. Since they’re ratios, they’re related.*  *Figure out the adjacent’s side length using the Pythagorean Theorem:* |
| |  |  | | --- | --- | |  |  | | Since ,  , which means:  Since we know , we  can use it to calculate | |

### 8.2.4. Improper Integrals

|  |  |
| --- | --- |
| Type 1: Infinite Bounds  When one of the integral’s bounds , | Type 2: Infinite  When at , |

For Type 1, it’s a convention in MAT137 to add at the front; MAT157 doesn’t do it. It’s not a big deal.

|  |  |
| --- | --- |
| eg. Evaluate | *Type 1 improper integral.*  *Since* |
|  |

|  |  |
| --- | --- |
| eg. Evaluate | *Ensure there’s one improper integral type per integral. This is why splits into and , where* |
|  |
|  | |

Sometimes, we can’t compute the integral, but we can test if it converges (ie. doesn’t diverge).

Let and be two continuous functions. Then we can apply:

|  |  |
| --- | --- |
| **Comparison Test** | **Limit Comparison Test** |
| If for , | If for , and , and  Then we can conclude and |

|  |  |
| --- | --- |
| eg. Show whether converges/diverges. | *Comparison Test, using:*  *We already found two questions above this one.* |
| As , it converges, thus converges. |

|  |  |
| --- | --- |
| eg. Show whether converges/diverges. | *Limit Comparison Test, with:*  *vs.*  *Note that I got rid of every term that wasn’t the function’s highest degree. The function on the right is much easier to calculate than the function on the left.* |
| |  |  | | --- | --- | |  | Since converges, converges. | |

P-Series

This is a very important class of functions common in sequences and series. For any

|  |  |  |
| --- | --- | --- |
|  |  | Converges *(derive from )* |
| Diverges *(derive from )* |

Note that when, the integral becomes , which diverges.

## 8.3. Application

### 8.3.1. Area

The area of a region under and above in interval is

* Nothing changes when or . The above two rules still hold true.

|  |  |
| --- | --- |
| eg. Find the area of all regions bounded by , , and . | *First find where the two main functions intersect so we know the upper/lower bounds.*  *x = 3 tells us upper limit is 3.*  *We’re finding true area, so in , the upper function must always be .* |
| In , . In , . |

### 8.3.2. Volume

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| If we take a 1D line of length *r*, |  | square it, |  | and multiply , we get a 2D circle: |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Think of this line as a very thin “slice” of 2D area , |  | square it, |  | and multiply for a “3D” shape: |  |

I’m not being “mathematically correct” here; and it’s not a proper integral. I just want you to intuitively compare a hypothetical infinitesimally-thin “slice” of 2D area to a 1D line.

What if we chose a and different integrals bounds?

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Let .  If we take the 2D area , |  | split it into infinitely many “slices” and square each slice, |  | and multiply , we get a cone: |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Let .  If we take the 2D area , |  | split it into infinitely many “slices” and square each slice, |  | and multiply , we get a cylinder: |  |

If we try calculating these integrals, we get:

* which is the volume of a cone.
* which is the volume of a cylinder.

|  |  |  |  |
| --- | --- | --- | --- |
| It thus follows that if we take the 2D area for any , |  | split it into infinite slices, square the slices, and multiply to the whole thing, we get the shape: |  |

We’re taking a 2D area and “rotating” it around the *x*-axis to produce a 3D shape, the **surface of revolution**. What if we rotated an area around the *y*-axis instead?

|  |  |  |  |
| --- | --- | --- | --- |
| We’d take our 2D area , |  | and rotate it around the *y*-axis to obtain the shape: |  |

The derivation for the formula of *y*-axis rotation is trickier to visualize.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Take a 1D line of length *h* at , |  | find the surface area of a cylinder at that position, |  | and get rid of the two circles’ areas. |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Think of this line as a very thin “slice” of 2D area , |  | find the surface area of a cylinder at that position, |  | and get rid of the two circles’ areas. |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Let .  If we take the 2D area , |  | split it into infinitely many “slices” and find surface areas, |  | The infinitely many surface areas “add up” to a volume. |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Let .  If we take the 2D area , |  | split it into infinitely many “slices” and find surface areas, |  | The infinitely many surface areas “add up” to a volume. |  |

In conclusion, these are the surface of revolutions of or on interval :

|  |  |
| --- | --- |
| ***x*-axis Rotation** | ***y*-axis Rotation** |
|  |  |
|  |  |

Some questions are defined such that you must write them like and thus use the alternate formulas.

This whole section about finding 3D volume in a 1D Calculus course is weird. My math prof said something like he thinks this content is here “because many people don’t go on to take MAT235/237/257, so the department needs to cram as much content as possible in 137 but also make it understandable.”

|  |  |
| --- | --- |
| eg. What is the volume of the 3D shape formed by rotating around the *y*-axis the region bounded by and ? | *(From MAT137, August exam 2017)*  *In questions like these that combine area with volume, treat as function on its own and substitute it into .* |
| |  |  | | --- | --- | | *On ,* |  | |

|  |  |
| --- | --- |
| eg. Derive the volume of a sphere using integration. | This is . |
| The equation for a circle of radius is . Isolate *y* to get  Thedomain of thus function is . Let’s revolve this around the *x*-axis.   |  |  | | --- | --- | |  |  | |

# 9. Sequences and Series

## 9.1. Sequences

### 9.1.1. Definition

A sequence is a set. It’s basically just like but with a domain of .

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
| ’s initial behaviour is irrelevant to .   * eg. , still converges at . | can be bounded above/below and still diverge.   * eg. |

|  |  |
| --- | --- |
| **Growth Rate Hierarchy** | *The a’s are unique; they can be any constant.*  *Note: In CS, . This has to do with a function’s running-time.* |
| This relates the convergence of functions: |

You may use the growth rate hierarchy for integrals, sequences, and series, but NOT for computing limits (otherwise, you might lose marks), but it is a handy tool to check your work.

|  |
| --- |
| eg. Find the infimum and supremum of |
| |  |  | | --- | --- | | You can do limit and show it converges to .  This oscillates endlessly between and . | The infimum of is  The supremum of is | |

|  |  |
| --- | --- |
| eg. The sequence is **eventually bounded above** when such that is bounded above. Prove that if a sequence is eventually bounded above, then it is bounded above. | *(From MAT137, April exam 2017)*  *I’m not sure if this answer is valid since I never did this type of proof in my class. But it makes sense.*  *Think of this kind of like EVT.*  *Expand the definition of "bounded above”. I used different variable names.* |
| Assume is eventually bounded above, meaning  Show is bounded above.  Since is a finite set of numbers, exists  Thus is bounded above by  We know is bounded above, meaning exists.  Since and  ,  Thus  is bounded above by |

### 9.1.2. Proving Convergence

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| eg. Prove the recursively-defined sequence converges to and evaluate . | *If you don’t get an answer here, the proof actually gets easier. Just prove that diverges.*    *The goal is to follow the statement*   |  |  | | --- | --- | | *S is bounded above*  *S is non-decreasing* | *S converges* |   *Prove the premises with induction*  *This logic is tricky to follow at first, but all other proofs will use this exact same logic.*  *Copy your rough work here.*  *Depending on S, you may need to follow this statement instead:*   |  |  | | --- | --- | | *S is bounded below*  *S is non-increasing* | *S converges* | |
| *Rough Work:*  *Let’s assume converges to , meaning*  *Since , ,* |
| Show is bounded above, meaning  Pick  Proof by induction, let  Base Case:  Inductive Step:  Assume  Show    Show is non-decreasing, meaning  Proof by induction, let  Base Case:    Inductive Step:  Assume  Show    Since is bounded above and non-decreasing, it converges at , meaning  Since *, ,* |

## 9.2. Series

### 9.2.1. Definition

|  |  |
| --- | --- |
| A series is the sum of a sequence. |  |
| It can also be thought of as a sequence of **partial sums** (ie. the sum of a finite part of a sequence) |  |

Just like with sequences,

* Initial behaviour doesn’t affect a series’ convergence.
* A series can diverge even if it is bounded.

|  |  |  |
| --- | --- | --- |
| If a series converges, we can do algebra on it: |  |  |

### 9.2.2. Convergence Tests

Usually, series are too difficult to compute; the focus is on testing them for whether they converge/ diverge. Different tests are ideal for different input classes. Examples for these tests are on next page.

|  |  |
| --- | --- |
| **Test** | **Description** |
| Zero Test |  |
| Integral Test  *(P-series)*  (*Of the form* ) | When , decreasing eventually, |
| Limit Comparison Test  *(Rational functions)* | When , both decreasing eventually, calculate. |
| Comparison Test  *(Trigonometric functions)*  *(Logarithmic functions)* | When , both decreasing eventually,   |  |  | | --- | --- | |  |  | |
| Ratio Test  *(Factorials)*  *(Exponentials)* | When , decreasing eventually, calculate |
| Absolute Convergence Test  *(Has negative terms)* |  |
| Alternating Series Test  *(Has alternating +/- terms)* | When , decreasing eventually, |

Every MAT137 exam will ask you to perform convergence tests. There are three possible outcomes:

* A series is **divergent** when diverges.
* A series is **conditionally convergent** whendivergesbut converges.
* A series is **absolutely convergent** whenconverges (which implies also converges).

To save doing extra work, test first unless you already know diverges.

* If converges, then is absolutely convergent
* If diverges, then test . If it converges, then is conditionally convergent

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Zero Test.* |
| **diverges** |

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Limit Comparison Test.*  *Pick to be simpler. Get rid of the secondary terms in the numerator and denominator. should be in the p-series.*  *On the exam, convergence test questions don’t require you to show any work.*  *Integral Test.*  *Having a constant in front of the p-series doesn’t change whether it converges/diverges* |
| Let                is part of the *p*-series, with , thus diverges  Since diverges, diverges  Since diverges, **diverges** |

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Special Case!*  *Integral Test on*  *I’m not checking the tests’ preconditions because as long as you have no trig in it, your function is probably positive and eventually decreasing.*  *Absolute Convergence Test* |
| Set , then and            Since converges, converges.  Thus is **absolutely convergent**. |

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Comparison Test, using the fact that .*  *Integral Test.*  *Absolute Convergence Test* |
| is part of the *p*-series, with , thus converges  Since converges, converges  Since converges, converges  Since converges, is **absolutely convergent**. |

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Integral Test.*  *diverges*  *Alternating Series Test*  *converges* |
| is part of the *p*-series, with , thus diverges  Since is of the form (where ),  Thus is **conditionally convergent.** |

My MAT137 prof used small angle approximation to get , but I feel it’s still kind of a logical leap from . I don’t really know why it follows, but okay.

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Ratio Test*  *Remember for factorials:*  *Thus it follows that*  *Absolute Convergence Test* |
| Thus is **absolutely convergent.** |

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Since the domain of is*  *AKA*  *I chose*  *Comparison Test*  *Limit Comparison Test*  *Integral Test* |
| *Rough Work:*  *When you spot a logarithm, ignore it for now. If the series still converges, then the series including the logarithm converges.*    *Do Limit Comparison Test:*    *Do Integral Test and then p-series, realize this converges. Now add the logarithm back.*    *Use the* ***Growth Rate Hierarchy****: . Do Comparison Test:*    *We need to choose a such that so we get a converging result in the p-series.* |
| Since .      Let                is part of the *p*-series, with , thus converges  Since converges, converges  Since converges, converges  Since converges, converges  Thus is **absolutely convergent**. |

|  |  |
| --- | --- |
| eg. Perform a convergence test on | *Since is on the denominator, if , then*  *Comparison Test*  *Limit Comparison Test*  *Integral Test* |
| *Rough Work:*  *Do Limit Comparison Test:*    *Do Integral Test and then p-series, realize this diverges. Now add the logarithm back and use the Growth Rate Hierarchy’s plus Comparison Test.*    *This means we need to choose a such that so we get a diverging p-series.* |
| Since .      Let                is part of the *p*-series, with , thus diverges  Since diverges, diverges  Since diverges, diverges  Since diverges, diverges |

Reminder that when proving only works when .

Reminder that when proving only works when .

### 9.2.3. Power Series

A power series with a **center of expansion** is a function with the form

The **radius of convergence** is a number where:

* When ,diverges.
* When , can converge or diverge (depends on the function).
* When , converges/is “analytic”
  + is infinitely differentiable in this region; always has the same
  + If converges everywhere, then

There are two ways to calculate .

|  |  |
| --- | --- |
| **Ratio Test** | **Root Test** |
|  |  |

The **interval of convergence** is the region of *x*-values where converges. It can be found by:

* *Slower Method:* Find with Ratio/Root Test, substitute it into and
* *Faster Method:* Calculate , solve for , and check cases when .

My MAT137 prof only taught us the faster method and barely even mentioned how to calculate and do the slower method, so don’t use it.

|  |  |
| --- | --- |
| eg. Find the radius and interval of convergence for | *This question type, plus convergence tests, are very common on exams,*  *When ,*  *You must perform a convergence test on and .* |
| When  Thus converges at  When ,  Do the same as the above, and you’ll find converges too.  The interval of convergence is |

### 9.2.4. Taylor Series

sucks to compute with. What if we instead wrote it as a polynomial?

Let . Let’s pick a point (eg. ), and try to create a polynomial by substituting:

*Degree 0 Polynomial:*

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

|  |  |
| --- | --- |
| We get a trash approximation of |  |

*Degree 1 Polynomial:*

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
| This isn’t stellar, but at least it’s getting a little bit closer to . |  |

*Degree 2 Polynomial:*

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |

I’m not showing another graph because this is still the same function as Degree 1.

*Degree 3 Polynomial:*

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

|  |  |
| --- | --- |
| is starting to wrap around . It’s becoming accurate in a growing area centered around . |  |

*Degree 4 Polynomial:*

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
| *Degree 5 Polynomial:*   |  |  |  |  | | --- | --- | --- | --- | |  |  |  |  | | | | |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

|  |  |
| --- | --- |
| continues wrapping around the curves of , most accurate around a **center of expansion**at . |  |

As we add more and more polynomials, becomes a power series. The area of accurate approximation (ie. interval of convergence) begins to eventually expand to all of .

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

is called the Taylor Series of : an infinite series of rising powers that is exactly equal to a function.

|  |  |
| --- | --- |
| The **Taylor Series** of with center of expansion is a type of power series | The Taylor Series of with center of expansion , AKA **Maclaurin Series**, is |

I recommend committing these to memory:

|  |  |  |
| --- | --- | --- |
| **Maclaurin Series** | **Expanded** | **Interval of Convergence** |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

A Taylor Series approximation is only accurate in its **symmetric** interval of convergence.

* Note that you probably saw the bottom series from high school: “ when ”
* Why ? Because , and Taylor series are symmetric, so it’s limited to .
* Note that a tangent line is a degree-1 polynomial Taylor Series approximation.

These operations can be performed on a Taylor Series (they don’t change the interval of convergence):

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
| eg. Find the Maclaurin Series of . | *If you’ve ever heard of* ***hyperbolas****, the function is called* ***hyperbolic sine****, or “sinh” for short. is “cosh”.*  *You can write any as . The same applies with the other series.*  *Check addition of series from 9.2.1.*  *Remember to include the interval of convergence!* |
|  |

|  |  |
| --- | --- |
| eg. Prove the Small Angle Approximation | *Substitute the Maclaurin Series*  *Only the first term, 1, has no multiple of x.* |
|  |

|  |  |
| --- | --- |
| eg. Prove Euler’s formula: . | *Don’t be scared! Keep calm and use Taylor Series.*  *The “…” notation is less formal, but we use it because otherwise, this is really hard to calculate.*  *This formula is fundamental, studying the complex plane:* |
|  |

|  |  |
| --- | --- |
| eg. Find the Maclaurin Series of . | *Deal with positive through double negative*  *Integral/derivative of both sides is a valid mathematical operation.*  *Recall is valid.*  *Recall . The isn’t a power of x, so it isn’t affected.*  *You MUST add C every time you integral, because the value of C is a single constant. It’s usually 0, but it might not be.* |
| *Substitute* |

|  |  |
| --- | --- |
| eg. Evaluate . | *Note the denominator. The only series of that form is .*  *Multiply by to mold the numerator into*  *Derivative both sides so that becomes*  *Divide by on both sides to mold into*  *Note that the derivative only affects powers of and stay the same.*  *The doesn’t match the that we need. So we choose such that , AKA , AKA*  *We need:*    *We have:*    *So it follows that* |
| Substitute |

|  |  |
| --- | --- |
| eg. Evaluate the infinite repeating decimal using Taylor series. | *Since*    *and*    *it follows that*    *and thus* |
| Since , |

|  |  |
| --- | --- |
| eg. Let . Evaluate . | *(From MAT137, April exam 2017)*  *I do not recommend differentiating 42 times.*  *Find the terms in the series of and of the format \_\_.*  *After differentiating 42 times, these terms will be the only constants. will equal those two constants.*  *Usually, you won’t have to simplify any further than this.* |
| After differentiating 42 times, the term becomes    After differentiating 42 times, the term becomes |

|  |  |
| --- | --- |
| eg. Find the Taylor Series expansion for at . | *(From MAT137, August exam 2017)*  If you differentiate past , everything is 0.  when . |
| |  |  | | --- | --- | |  |  | |

|  |
| --- |
| eg. Find the Taylor Series expansion for at . |
| The Maclaurin Series for is .  This makes sense as for all . We know for all , thus |

### 9.2.5. Telescoping Series

A telescoping series is a type of series where all adjacent terms cancel out. Set .

|  |
| --- |
| eg. Evaluate |
|  |

# 10. Resources

[MAT137 Playlist](https://www.youtube.com/channel/UCLzpR8AiHx9h_-yt2fAxd_A/playlists?view=50&sort=dd&shelf_id=1)

Videos of all MAT137 topics by the late Alfonso Gracia-Saz, who taught the course for a long time. These videos are clear and most importantly, concise.

[MAT137 Lecture Notes](http://home.tykenho.com/LectureNotes137_Preview.pdf)

A MAT137 textbook by Tyler Holden. It’s clear and contains detailed explanations, but I believe he teaches in UTM and some covered topics are not as relevant/big a focus.

MAT137 [Past Practice Questions](http://www.math.toronto.edu/~alfonso/137/137.html?videos) and [Assignments](http://www.math.toronto.edu/~alfonso/137/137.html?ps) (with Answers!)

These questions can be difficult, but serve as good exam preparation.

[Old Exam Repository](https://login.library.utoronto.ca/index.php?url=https://exams.library.utoronto.ca/)

Contains past MAT137 exams. It requires you have a UTORid to login. All questions from Artsci MAT137 exams are already in *Questions from MAT137*.

[Essence of Calculus](https://www.youtube.com/playlist?list=PLZHQObOWTQDMsr9K-rj53DwVRMYO3t5Yr)

An excellent Youtube playlist by 3Blue1Brown with very pretty illustrations/animations of calculus concepts. Builds great visual intuition for concepts like:

* Epsilon-Delta Definition of Limit
* Definition of Integral
* Definition of Taylor Series
* Derivative Techniques (Product Rule, Chain Rule, Implicit Differentiation)